Exact solutions for steady two-dimensional flow of a stratified fluid

By CHIA-SHUN YIH
Department of Engineering Mechanics, University of Michigan†

(Received 15 February 1960)

Three classes of exact solutions for steady two-dimensional flows of a stratified fluid are found. The flows which correspond to these solutions have arbitrary amplitude (however defined). Two of the three classes of solutions have close bearings on the lee-wave problem in meteorology. It is also shown that the amplitudes of the lee-wave components (if there is more than one component) depend not on the details of the shape of the barrier, but only on certain simple integral properties of the function for the singularity distribution generating the barrier.

1. The equation governing steady two-dimensional flow of a stratified fluid

This study is restricted to steady two-dimensional flows of a stratified fluid, assumed incompressible, inviscid, and non-diffusive. For such flows the Euler equations are

$$\rho\left(u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right)(u,w) = -\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)p-(0,g\rho), \tag{1}$$

in which p and ρ are the pressure and the density, g is the gravitational acceleration, x and z are Cartesian co-ordinates, with z measured in the direction opposite to that of gravity, and u and w are the velocity components in the directions of increasing x and z, respectively. Since the fluid is incompressible and non-diffusive and the flow is steady,

$$\left(u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right)\rho = 0. \tag{2}$$

This permits the equation of continuity to be written in the form

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z} = 0, \tag{3}$$

and the equations of motion to be written as

$$\rho_0\left(u'\frac{\partial}{\partial x}+w'\frac{\partial}{\partial z}\right)(u',w') = -\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right)p-(0,g\rho), \tag{4}$$

in which
$$u' = u(\rho/\rho_0)^{\frac{1}{2}}, \quad w' = w(\rho/\rho_0)^{\frac{1}{2}}, \tag{5}$$

† At Department of Applied Mathematics and Theoretical Physics, University of Cambridge, during 1959–60.

and ρ_0 is a reference density. With the pseudo-vorticity ω' defined by

$$\omega' = \frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z},$$

and ψ' the pseudo stream function, so that

$$u' = \frac{\partial \psi'}{\partial z}, \quad w' = -\frac{\partial \psi'}{\partial x}, \tag{6}$$

the equations of motion can be further simplified to

$$-\rho_0 \omega' \frac{\partial \psi'}{\partial x} = \frac{\partial}{\partial x}[p + \tfrac{1}{2}\rho_0(u'^2 + w'^2)], \tag{7}$$

$$-\rho_0 \omega' \frac{\partial \psi'}{\partial z} = \frac{\partial}{\partial z}[p + \tfrac{1}{2}\rho_0(u'^2 + w'^2)] + g\rho. \tag{8}$$

If (7) is multiplied by dx and (8) by dz, and the results added,

$$-\rho_0 \omega' d\psi = d[p + \tfrac{1}{2}\rho(u^2 + w^2)] + g\rho\, dz = dH - gz\, d\rho, \tag{9}$$

in which

$$H = p + \tfrac{1}{2}\rho(u^2 + w^2) + g\rho z$$

is the Bernoulli constant, which is constant along a streamline but may vary from streamline to streamline, and is hence a function of ψ' alone. Since for steady flows streamlines are path-lines, and for an incompressible and non-diffusive fluid ρ is constant along a path-line, ρ is also a function of ψ' alone. Thus (9) can be written as

$$\rho_0 \nabla^2 \psi' + gz \frac{d\rho}{d\psi'} = \frac{dH(\psi')}{d\psi'} = h(\psi'), \tag{10}$$

with $\nabla^2 \psi'$ replacing $-\omega'$. Equation (10) is the governing equation sought, and is a modified form of an equation due to Long (1953), who did not relate the arbitrary function of ψ' on the right-hand side of his equation to the Bernoulli constant.

2. Types of exact solutions

To discover exact solutions of (10), it is natural to consider circumstances in which the equation becomes linear in ψ. To this end, one may adopt two different approaches. Since the function $h(\psi')$ is related to the upstream condition, one may either try different upstream conditions and see whether these will make the equation linear, or assume the equation to be linear to start with and inquire what the corresponding upstream condition must be. The second approach is evidently both exhaustive and more economical. Adopting the first approach, Long said in his impressive paper (Long 1953) of the case in which ρu^2 is constant far upstream where the flow is parallel: 'This is the only case I have been able to discover for which the differential equation governing the motion of a stratified fluid is exactly linear.' This is hardly surprising, because the equation originally discovered by Long is in the form

$$\nabla^2 \psi + \frac{1}{\rho} \frac{d\rho}{d\psi} \left(\frac{\psi_x^2 + \psi_y^2}{2} + gz \right) = f(\psi), \tag{10a}$$

in which ψ is the usual stream function and subscripts indicate partial differentiation, and in this form it is quite unsuitable for discovering all the cases in which (10a) is exactly linear. The transformation (6) takes care of the inertia effect of density variation once and for all, removes the troublesome terms (representing the inertia effect) in Long's equation, and produces (10), which, while equivalent to Long's equation, is so much simpler that the second approach can now be applied. It will be shown by the second approach that there are three essentially distinct classes of flows for which (10) is exactly linear, each consisting of infinitely many flows. One of these three classes contains Long's case as a special (but very important) case. Thus the simple transformation (6) proves to be a very fruitful one.

For (10) to be linear in ψ, $d\rho/d\psi'$ and $h(\psi')$ must be linear in ψ'. The linear equation therefore has the general form

$$\nabla^2\psi' + gz(C + C_1\psi') = C_2 + C_3\psi'. \tag{11}$$

If $C_1 = C_3 = 0$, this equation has the form

$$\nabla^2\psi' + Cgz = C_2. \tag{11a}$$

If $C_1 = 0$ but $C_3 \neq 0$, ψ' can be changed by a constant, and (11) becomes

$$\nabla^2\psi' + Cgz = C_3\psi'. \tag{11b}$$

The class of flows governed by this equation includes Long's case as a special case. If $C_1 \neq 0$ but $C_3 = 0$, (11) becomes, after ψ' has been changed by a constant,

$$\nabla^2\psi' + C_1gz\psi' = C_2. \tag{11c}$$

If C_1 and C_3 are both different from zero, the origin of z can be shifted so that the resulting equation is

$$\nabla^2\psi' + gz(C + C_1\psi') = C_0,$$

which, on changing ψ' by a constant, becomes identical in form with (11c).

If now the dimensionless parameters

$$\xi = x/d, \quad \eta = z/d, \quad \Psi = \psi'/Vd$$

are introduced, in which d is a reference length and V a reference velocity, (11a) to (11c) assume the form

$$\nabla^2\Psi + A\eta = B, \tag{12a}$$

$$\nabla^2\Psi + A\eta = B\Psi, \tag{12b}$$

$$\nabla^2\Psi + A\eta\Psi = B, \tag{12c}$$

in which A and B are dimensionless constants, and, now and henceforth,

$$\nabla^2 \equiv \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

3. Class (a): pseudo-potential flows

For class (a), the general solution is of the form

$$\Psi = \Psi_h - A\eta\begin{Bmatrix}\frac{1}{6}\eta^2\\ \frac{1}{2}\xi^2\end{Bmatrix} + \tfrac{1}{2}B\begin{Bmatrix}\eta^2\\ \xi^2\end{Bmatrix} + C\xi\eta + D\begin{Bmatrix}\eta\\ \xi\end{Bmatrix}, \tag{13}$$

in which a fifth constant has been suppressed because Ψ can be changed by an arbitrary constant. Either member of the brackets in (13) can be used, or a linear combination of both members, except that for the first two brackets the linear combinations must be such that (12*a*) is satisfied. The first member on the right-hand side of (13) is a harmonic function satisfying the Laplace equation

$$\nabla^2\Psi_h = 0. \tag{14}$$

A re-examination of (10), (11*a*) and (12*a*) reveals that

$$A = \frac{1}{\rho_0}\frac{d\rho}{d\Psi}\frac{gd}{V^2}, \tag{15}$$

so that, once Ψ is determined,

$$\rho = \frac{A\rho_0 V^2}{gd}\Psi + \text{constant} \tag{16}$$

is also known up to an additive constant. Furthermore, if ρ decreases with increasing η (stable stratification), and the flow is from left to right, $d\rho/d\Psi$ is negative, and the expression for A suggests that it is really the negative of the reciprocal of the square of a Froude number:

$$A = -F^{-2}. \tag{17}$$

A first example is furnished by a stratified flow between two horizontal boundaries, at $z = 0$ and $z = d$, into a line sink whose trace is situated at the origin. Since the flow is symmetric with respect to the z-axis, it is convenient to consider the flow to be separated by a plane vertical wall at $x = 0$ into two mirror images. The left one of these will be considered. If the flow at $x = -\infty$ is parallel and with the parabolic distribution for the velocity (as weighted by the factor $(\rho/\rho_0)^{\frac{1}{2}}$)

$$U(\eta) = 6U_m\eta(1-\eta), \tag{18}$$

in which U_m is the mean of U, then with U_m as the representative velocity V,

$$A = 12, \quad B = 6, \quad C = D = 0$$

in (13), and

$$\Psi = \Psi_h - 2\eta^3 + 3\eta^2. \tag{19}$$

The boundary conditions are $\Psi = 0$ at $\eta = 0$, $\Psi = 1$ at $\eta = 1$, and at $\xi = 0$ $(0 < \eta \leqslant 1)$; $\Psi = 3\eta^2 - 2\eta^3$ at $\xi = -\infty$. In terms of Ψ_h, these are

$$\Psi_h = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1, \tag{20}$$

$$\Psi_h = 1 + 2\eta^3 - 3\eta^2 \quad \text{at} \quad \xi = 0 \quad (0 < \eta \leqslant 1), \tag{21}$$

$$\Psi_h = 0 \quad \text{at} \quad \xi = -\infty. \tag{22}$$

By the method of separation of variables the solution of (13) to (16) is found to be

$$\Psi_h = \Sigma A_n e^{n\pi\xi} \sin n\pi\eta, \tag{23}$$

in which

$$A_1 = \frac{2}{\pi}, \quad A_2 = \frac{1}{\pi}\left(1 + \frac{3}{\pi^2}\right), \quad A_3 = \frac{2}{3\pi}, \quad A_4 = \frac{1}{\pi}\left(\frac{1}{2} + \frac{3}{8\pi^2}\right), \quad A_5 = \frac{2}{5\pi}, \quad \text{etc.}$$

The final solution is

$$\psi' = U_m d(\Psi + 3\eta^2 - 2\eta^3), \tag{24}$$

and the density variation at infinity is

$$\rho = \rho_0 + (\rho_0 - \rho_1)(\Psi + 3\eta^2 - 2\eta^3), \tag{25}$$

in which ρ_0 is the density at the bottom plate and ρ_1 that at the top plate. The flow pattern can be expected to have an eddy at the corner bounded by the upper boundary and the vertical wall, extending to infinity.

For another example consider the case of stratified flow with $\Psi = -\frac{1}{6}A\eta^3$ at ∞. The lower boundary consists of a semicircle ($r = r_0$, $0 \leqslant \theta \leqslant \pi$) and the lines ($\theta = 0$, $\theta = \pi$, $r \geqslant r_0$). No upper boundary is imposed. After the solution is obtained, any streamline (and in particular the one on which ρ is zero) can be taken to be the upper boundary. Here $C = 0$ in (13), and the upper members in the brackets of (13) can be taken. For illustration, B and D will be taken to be zero. The general case in which B and D are not equal to zero can be solved similarly. The reference length is now the radius of the cylinder, r_0. For con-

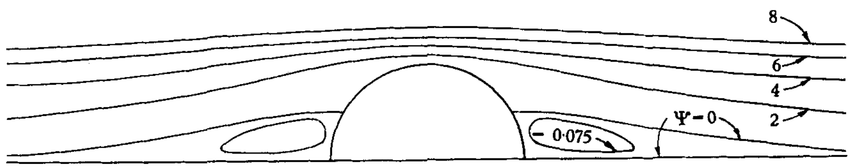

FIGURE 1. Pattern of a pseudo-potential flow of a stratified fluid over a semicircle.
$\Psi = -\dfrac{24\psi'}{Cgr_0^3}$, $C = \dfrac{1}{\rho_0}\dfrac{d\rho}{d\psi'}$, r_0 = radius of semi-circle, $r_1 = r/r_0$.

venience, let r/r_0 be denoted by r_1. The boundary conditions are (for $\Psi = -\frac{1}{6}A\eta^3$ at infinity): $\Psi_h = 0$ at $\eta = 0$ (at least for $r \geqslant 1$), $\Psi_h = 0$ at $r_1 = \infty$, $\Psi_h = \frac{1}{6}A\eta^3$ for $r_1 = 1$ ($\eta \geqslant 0$). The solution is

$$\begin{aligned}\psi' &= \frac{Ar_0^3}{24}\left(\frac{3\sin\theta}{r_1} - \frac{\sin 3\theta}{r_1^3} - 4r_1^3\sin^3\theta\right)\\ &= \frac{Ar_0^3}{24}\left[3\left(\frac{1}{r_1} - r_1^3\right)\sin\theta - \left(\frac{1}{r_1^3} - r_1^3\right)\sin 3\theta\right].\end{aligned} \tag{26}$$

The flow pattern is shown in figure 1. As can be readily calculated from (20), the stagnation points are the points:

$$(r,\theta) = (\infty, 0),\ (\infty, \pi),\ (r_0, 0),\ (r_0, \pi),\ (r_0, \tfrac{1}{6}\pi),\ (r_0, \tfrac{5}{6}\pi).$$

These form the corners of two roughly triangular eddies, symmetrically located over the horizontal boundary, as shown in the figure.

4. Class (*b*): waves of arbitrary amplitude, with application to flow over a barrier

The class of flows governed by (12*b*) can be applied to the atmosphere. If the flow is parallel, the corresponding stream function is governed by the equation

$$\frac{d^2\Psi_1}{d\eta^2} + A\eta = B\Psi_1, \tag{27}$$

which has the solution

$$\Psi_1 = \frac{A}{B}\eta + C\sinh\sqrt{B}\,\eta + D\cosh\sqrt{B}\,\eta. \tag{28}$$

If the solution of (12*b*) is written as the sum of two parts, i.e.

$$\Psi = \Psi_1 + \Psi_2, \tag{29}$$

then Ψ_2 satisfies

$$\nabla^2\Psi_2 - B\Psi_2 = 0, \tag{30}$$

the solution of which satisfying the boundary conditions† $\Psi_2 = 0$ at $\eta = 0$ and $\eta = 1$ is of the form

$$\Psi_2 = \sum_{n=1}^{\infty} A_n \exp \pm (B + n^2\pi^2)^{\frac{1}{2}}\,\xi \sin n\pi\eta. \tag{31}$$

The special case $C = D = 0$ has been considered by Long (1953). In this case, if Ψ and Ψ_1 are expressed in terms of Ud, with U denoting the *uniform* velocity (weighted by the factor $(\rho/\rho_0)^{\frac{1}{2}}$) far upstream, $\Psi = \Psi_1 = \eta$ far upstream, and $A = B$. Thus (17), (28), (29) and (31) produce the solution

$$\Psi = \eta + \sum_{n=1}^{\infty} A_n \exp\{\pm (n^2\pi^2 - F^{-2})^{\frac{1}{2}}\,\xi\} \sin n\pi\eta. \tag{32}$$

The solution by Long (1955) for stratified flow over barriers and the solution by Yih (1958) for stratified flow into a sink are of the form (32). These solutions can now be generalized to apply to the infinitely many upstream conditions described by (28). This gain in generality has been possible because (10) is substantially simpler than (10*a*). The actual modification of the solutions of Long and of Yih for application to the generalized upstream condition is straightforward and will not be presented here. Suffice it to say that (28) possesses so much latitude that an actual upstream wind condition can be much better approximated by (28) than by $\Psi_1 = \eta$, by appropriate choice of the constants A, B, C and D. Thus the meteorological significance of (27) is somewhat enhanced. Care must be taken, however, to ensure that Ψ (or Ψ_1, since Ψ_2 is assumed to vanish far upstream) be monotone in η, for otherwise (since $d\rho/d\Psi_1$ is constant) an unstable density distribution would be present in a part of the fluid in parallel flow.

If B is negative and greater in numerical value than $(N\pi)^2$ but less than $(N+1)^2\pi^2$, Ψ_2 in (31) will contain N terms periodic in ξ, representing wave motion. It is commonly assumed (and the assumption has been experimentally verified) that upstream waves do not occur. But if we do not inquire how the waves are made and only ask whether a periodic condition can be consistent with (27), we see immediately that the answer is in the affirmative, because

$$\Psi = \Psi_1 + \sum_{n=1}^{N} (B_n \cos|B + n^2\pi^2|^{\frac{1}{2}}\,\xi + C_n \sin|B + n^2\pi^2|^{\frac{1}{2}}\,\xi) \sin n\pi\eta \tag{33}$$

† The condition $\Psi_2 = 0$ at $\eta = 0$ is imposed to make $\eta = 0$ a streamline. This is desirable if the ground surface is flat or at least flat as $\xi \to \infty$ (since lee-waves, if they exist, will not die out as $\xi \to \infty$). If there is a barrier on the surface of which we demand $\Psi = 0$, then the streamline $\Psi = 0$ consists of the ground surface *and* the line $\eta = 0$ (which may constitute part of the ground surface), if the condition $\Psi_1(0) = 0$ is imposed. The boundary condition $\Psi_2 = 0$ at $\eta = 1$ follows from the requirement $\Psi =$ constant at $\eta = 1$. A discussion of the realism of this requirement when the theory is applied to the atmosphere will be given later in this paper.

is a solution of (27), with Ψ_1 given in (28). The corresponding density distribution given by

$$\frac{d\rho}{d\Psi} = \text{constant} \tag{34}$$

is not necessarily unstable, even though in certain regions the density increases upwards, because of the presence of acceleration toward the centre of curvature. The wave motion represented by (33) can have arbitrary amplitudes. For the particular case of $A = B$, $C = D = 0$,

$$\Psi = \eta + \sum_{n=1}^{N} (B_n \cos |F^{-2} - n^2\pi^2|^{\frac{1}{2}} \xi + C_n \sin |F^{-2} - n^2\pi^2|^{\frac{1}{2}} \xi) \sin n\pi\eta, \tag{35}$$

with
$$N\pi < F^{-1} < (N+1)\pi, \tag{36}$$

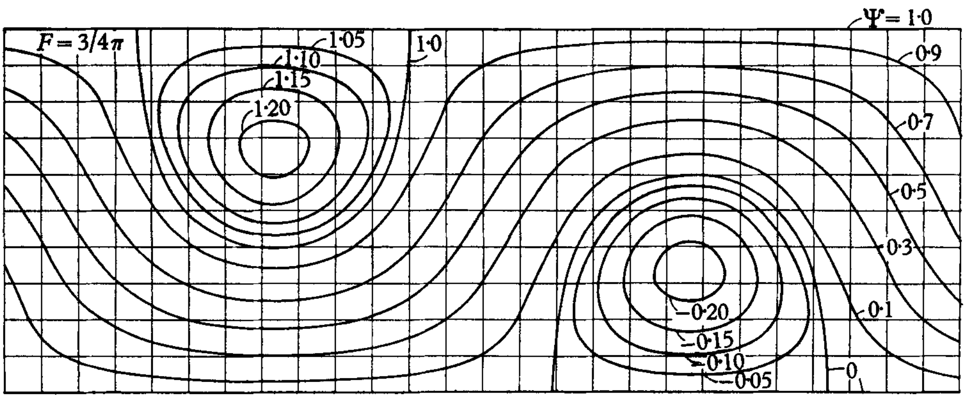

FIGURE 2. Stationary waves of arbitrary amplitude.
$$F = \frac{3}{4\pi}, \quad \Psi = -\frac{\psi'}{F^2 d^3 g C}, \quad C = \frac{1}{\rho_0}\frac{d\rho}{d\psi'}.$$

represents a period motion with N wave components. In spite of what Long himself said (as quoted in § 2), the waves represented by (35)—but not those represented by (33)—can be considered to have been discovered by him, because they are identical with the lee-waves he studied. The flow pattern for

$$\Psi = \eta + \frac{2}{\pi} \cos (F^{-2} - \pi^2)^{\frac{1}{2}} \xi \sin n\pi\eta, \tag{37}$$

with $F = 3/4\pi$, is shown in figure 2. The term η in (35) and (37) represents only a pseudo-uniform velocity field, because Ψ is a stream function for the flow field (u', w'), and not for the flow (u, w). Therefore the parallel flow represented by η does not correspond to an actually uniform velocity, and it is not possible by a shifting of co-ordinate axes to remove the parallel flow altogether. However, if drift is allowed, the discharge relative to a moving frame of reference can be made to be zero. The speed c with which the frame moves is exactly the phase velocity of the wave pattern relative to the moving frame, which is quite different from the stationary wave pattern. This speed can be calculated readily, and will not be given explicitly here.

Use of vorticity distributions for generating stratified flows over a barrier

For the case $A = B$, $C = D = 0$, Long (1955) gave a method for generating the solution for flow over a barrier which is approximately the same as one with a prescribed form. We shall now give an alternate method, for generating solutions of (12*b*) for flow over a barrier of unspecified (except implicitly) form. This method has three advantages over Long's method. First, it is simpler. Secondly, the determination of the coefficients is exact, and does not involve the solution of infinitely many equations containing infinitely many unknowns. Thirdly, by means of it we can prove the very important fact† that the amplitudes of the various lee-wave components depend not on the *details* of the shape of the barrier, but only on certain integral properties of the singularities generating the barrier. On the other hand, the shape of the barrier is directly though only approximately accounted for by Long's method, whereas the alternative method to be presented is entirely an inverse method which does not attempt to generate a solution for a *prescribed* barrier even approximately, except indirectly.

The method will now be described. When

$$(N\pi)^2 < -B < (N+1)^2\pi^2,$$

the solution of (12*b*) can be put in the form

$$\left.\begin{aligned} \Psi_- &= \Psi_1 + \sum_{N+1}^{\infty} A_n e^{a_n\xi} \sin n\pi\eta \quad (\text{for } \xi \leqslant 0), \\ \Psi_+ &= \Psi_1 + \sum_{n=1}^{N} (B_n \cos a_n \xi + C_n \sin a_n \xi) \sin n\pi\eta \\ &\quad + \sum_{N+1}^{\infty} D_n e^{-a_n\xi} \sin n\pi\eta \quad (\text{for } \xi \geqslant 0), \end{aligned}\right\} \tag{38}$$

in which Ψ_1 is given by (28) and

$$a_n = |B + n^2\pi^2|^{\frac{1}{2}}.$$

The coefficients A_n, B_n, C_n and D_n are determined by demanding

$$\Psi_- = \Psi_+ \quad \text{at} \quad \xi = 0, \tag{39}$$

$$\frac{\partial \Psi_-}{\partial \xi} - \frac{\partial \Psi_+}{\partial \xi} = f(\eta) \quad \text{at} \quad \xi = 0, \tag{40}$$

in which $$f(\eta) = 0 \quad \text{for} \quad \eta \geqslant a \quad \text{and} \quad \eta = 0, \tag{41}$$

and is arbitrary elsewhere. Since Ψ_- and Ψ_+ satisfy (12*b*), (39) and (40) ensure that Ψ_+ is the analytic continuation of Ψ_-. There are no singularities in the domain outside of the barrier (which is determined *a posteriori*), and there are no upstream waves. The function $f(\eta)$ corresponds to a sheet of distributed vortices (with horizontal axes) of variable strength at $\xi = 0$, extending from $\eta = 0$ to $\eta = a$. It determines implicitly the shape of the barrier.

† The author is indebted to Dr G. K. Batchelor for pointing out the possibility of this fact.

It will now be seen that B_n and C_n depend not on the details of the function $f(\eta)$ but only on certain of its integral properties, or, more precisely, on certain of its Fourier coefficients associated with the functions $\sin n\pi\eta$. Indeed, (39) demands that

$$A_n = D_n \quad (n > N), \tag{42}$$

$$B_n = 0 \quad (n \leqslant N), \tag{43}$$

and (40) demands

$$a_n(A_n + D_n) = 2\int_0^1 f(\eta)\sin n\pi\eta\, d\eta \quad (n > N), \tag{44}$$

$$a_n C_n = -2\int_0^1 f(\eta)\sin n\pi\eta\, d\eta \quad (n \leqslant N). \tag{45}$$

Thus A_n and D_n are determined from (42) and (44), and C_n from (45).† But (45) is most significant. It states that the amplitudes of the N wave components are equal to the first N Fourier coefficients of the function divided by a_n (which depends only on B and n, and is quite independent of $f(\eta)$). Certainly there are infinitely many functions which satisfy (41), have the same first N Fourier coefficients, and yet are different. The barriers corresponding to all these different generating functions have lee-waves with the same wavelengths and the same amplitudes, provided B is the same for the upstream flow. The situation is even independent of the coefficients A, C, and D in (28), though the shapes of the resulting barriers are dependent upon them. Near the barrier, the flow depends also on A_n and D_n, and hence also on the rest of the Fourier components. In other words, near the barrier the flow depends on all the Fourier coefficients of or on the full details of $f(\eta)$, as is to be expected. Thus we have arrived at a sort of St Venant's principle in stratified flow.

Now that the alternative method has proved fruitful, it is desirable to improve the method, in order to obtain some flexibility for dealing with barriers of specified forms. The method outlined above is good for constructing flows over rather bluff barriers, and is not adequate if the barrier is elongated. To remedy this situation, the obvious thing to do is to achieve a freedom for displacing the vortices (represented by $f(\eta)$) in the x-direction. This can be done simply by using two or more vertical vortex sheets at different values of x or ξ. If (38) is rewritten as

$$\Psi = \Psi_1 + \Psi_2,$$

in which the expressions for Ψ_2 are different for $\xi > 0$ and for $\xi < 0$, we see that Ψ_2 is the stream function which owes its existence to the presence of the barrier or of the singularities generating this barrier. If an additional line of singularities is situated at $\xi = b$, with distribution function $f_1(\eta)$, and still another situated at $\xi = c$, with distribution function $f_2(\eta)$, the solution is of the form

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4,$$

in which Ψ_3 and Ψ_4 are similar to Ψ_2 given in (38), except that the ξ in Ψ_2 should be changed to $\xi - b$ and $\xi - c$ for Ψ_3 and Ψ_4, respectively, and the coefficients are now determined from $f_1(\eta)$ for Ψ_3, and from $f_2(\eta)$ for Ψ_4. Generalization to

† If $B = -N^2\pi^2$, $a_N = 0$, and in order for the method to work the Nth Fourier coefficient of $f(\eta)$ must be zero.

the case of more than three vortex sheets is obvious. If it is desirable to use isolated vortices, we can simply take $f(\eta)$ to be a Dirac delta-function located somewhere above $\eta = 0$.

Use of sources, sinks, and doublets for generating stratified flows over a barrier

Instead of demanding (39) and (40), we can demand

$$\Psi_- - \Psi_+ = f(\eta) \quad \text{at} \quad \xi = 0,$$

$$\frac{\partial \Psi_-}{\partial \xi} - \frac{\partial \Psi_+}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0,$$

with

$$f(\eta) = 0 \quad \text{for} \quad \eta \geqslant a \quad \text{and} \quad \eta = 0,$$

and $f(\eta)$ arbitrary elsewhere. The function $f(\eta)$ now corresponds to a source distribution along a line element at $\xi = 0$. The solution is again in the form of (38), but the formulas for the coefficients are now

$$A_n = -D_n \quad (n > N), \quad C_n = 0 \quad (n \leqslant N),$$

$$A_n - D_n = 2\int_0^1 f(\eta) \sin n\pi\eta \, d\eta \quad (n > N),$$

$$B_n = -2\int_0^1 f(\eta) \sin n\pi\eta \, d\eta \quad (n \leqslant N).$$

If there is more than one line source, the generalization is the same as given in the last paragraph. The total algebraic sum of the sources must be zero in order that the barrier be closed.

Again, by taking $f(\eta)$ to be a Dirac delta-function, the solution corresponding to an isolated source or sink can be obtained. It can be readily shown that if a source is located at $\xi = -b$ and a sink of equal strength (m) is located at $\xi = b$ and at the same height h_s (dimensionless), the nth ($n \leqslant N$) lee-wave is represented by

$$-4m \sin n\pi h_s \sin a_n b \sin a_n \xi \sin n\pi\eta.$$

If b is small, the amplitudes of the lee-waves are proportional to $2mb$, which is the negative of the moment of the source and sink. Thus, for a doublet of strength μ (equivalent to $-2mb$), the amplitude of the nth lee-wave is

$$2\mu a_n \sin n\pi h_s,$$

and, for a doublet distribution from $\eta = 0$ to $\eta = a$, the amplitude of the nth lee-wave is

$$2a_n \int_0^a \mu(\eta) \sin n\pi\eta \, d\eta.$$

If the doublet (isolated or distributed vertically) is not located at $\xi = 0$, *only* the *phase* of the pertaining lee-waves will be changed (by an amount equal to the ξ-coordinate of the doublet or doublet distribution).

An example, and a discussion of the effect of the upper boundary

Figure 3 shows a stratified flow (case (*b*)) over a barrier, with waves in the lee. The velocity distribution far upstream is given indirectly by (28), with $C = D = 0$ and $A = B = -9/16\pi^2$, so that u' is constant far upstream. The flow is analytically given by (38) and (42) to (45), with

$$-f(\eta) = 10\sin 5\pi\eta \quad \text{for} \quad 0 \leqslant \eta \leqslant 0{\cdot}2,$$
$$= 0 \quad \text{for} \quad 0{\cdot}2 < \eta \leqslant 1.$$

There is a single lee-wave component, with wavelength $(6/\sqrt{7})d$, d being the depth from the (flat) upper boundary to the flat part of the lower boundary.

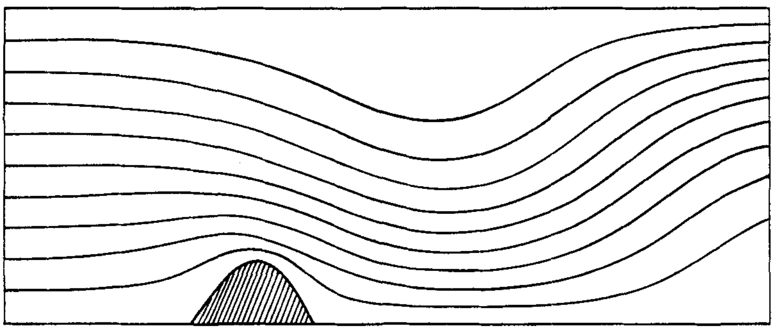

FIGURE 3. A stratified flow over a barrier, with waves in the lee.

Since the wave numbers of the lee-waves are a_n (for those n's that make $B+n^2\pi^2$ negative), the wavelengths must depend on B. In the example just given, $B = -F^{-2}$, with

$$F^2 = \frac{\rho_0 U'^2}{gd^2(d\rho/dz)},$$

U' being the (constant) velocity u' at $x = -\infty$, where $d\rho/dz$ is taken. Thus for a given U' and density distribution, the wavelengths depend on d. Consequently, the location of the upper boundary has in this case, as indeed it does in general, a profound influence on the flow pattern. This is not surprising, for as the depth increases the wave numbers for those lee-waves just able to stay stationary (i.e. to withstand the sweeping action of the parallel flow) must also increase, and vice versa, because the wave velocities increase with d and decrease with the wave-numbers (Yih 1960). However, the importance of the location of the imaginary upper boundary does raise the question of where to impose it in any given situation, and the question of the error committed by imposing it.

A partial answer to these questions can be obtained by considering two superposed layers of fluids. The lower layer has depth d, and is bounded below by a rigid plane boundary (the ground). The upper layer has depth d_u and is bounded above by an auxiliary rigid plane. This auxiliary boundary (not the one the effect of which is under discussion) is imposed only for convenience, and is not necessary, for the same conclusions can be reached on the assumption that the depth of the upper layer is infinite. The interface of the two layers is the place where the artificial rigid boundary under discussion is supposed to be imposed in the preceding analysis, and it is proposed to see under what conditions the presence of this boundary introduces only small errors. Let the density gradients in the two layers be constant but different, and the density be continuous at the interface. At $x = -\infty$, u' is supposed to be the same for both layers, and is denoted by U'. The equations governing fluid motion in the two layers are

$$\nabla^2\Psi - F^{-2}\eta = -F^{-2}\Psi, \quad \nabla^2\Psi_u - F_u^{-2}\eta = -F_u^2\Psi,$$

in which

$$F = -\frac{\rho_0 U'^2}{gd^2(d\rho/dz)}, \quad F_u = -\frac{\rho_0 U'^2}{gd^2(d\rho/dz)_u},$$

the density gradients being taken in the absence of waves. The parallel flow corresponds to

$$\Psi_1 = (\Psi_u)_1 = \eta,$$

and the wave motion is governed by (with the subscript 2 on the stream functions dropped for convenience)

$$\nabla^2\Psi = -F^{-2}\Psi, \quad \nabla^2\Psi_u = -F_u^{-2}\Psi_u.$$

These equations are to be solved with the boundary conditions

(i) $\Psi = 0$ at $\eta = 0$;

(ii) $\Psi_u = 0$ at $\eta = d_u/d$;

(iii) that the displacement be the same for both layers at the interface;

(iv) that the pressure (or, equivalently, the velocity) be continuous at the interface.

The solution of the eigenvalue problem so defined on the basis of a linearized analysis yields the secular equation

$$(F^{-2}-m^2)^{\frac{1}{2}}\cot(F^{-2}-m^2)^{\frac{1}{2}} = -(F_u^{-2}-m^2)^{\frac{1}{2}}\cot(F_u^{-2}-m^2)^{\frac{1}{2}}\left(\frac{d_u}{d}-1\right)$$

for the determination of the eigenvalues for the wave-number m in the factor $\sin m\xi$ contained in Ψ and Ψ_u. Since in the atmosphere the gradient of the potential density in the stratosphere is greater than that in the troposphere,

$$F_u^{-2} = \alpha F^{-2} \quad (\alpha > 1).$$

Hence

$$\frac{F_u^{-2}-m^2}{F^{-2}-m^2} = \frac{(\alpha-1)F^{-2}}{F^{-2}-m^2}+1.$$

In the atmosphere, if d is taken to be the depth of the troposphere, $F^2 \ll 1$. Consequently, if m is large enough to make $F^{-2}-m^2$ much smaller than F^{-2}, the secular equation can be replaced by

$$\tan(F^{-2}-m^2)^{\frac{1}{2}}\cot(F_u^{-2}-m^2)^{\frac{1}{2}}\left(\frac{d_u}{d}-1\right) = 0.$$

One set of solutions of this equation is

$$m = (F^{-2}-n^2\pi^2)^{\frac{1}{2}} = a_n,$$

so long as n is not so large an integer as to make m small. Another set of solutions is obtained by setting the other factor (in the approximate secular equation) equal to zero. The wave-numbers obtained are for waves in the upper layer when the interface is replaced by a rigid boundary. Thus, if we take d to be the depth of the troposphere and apply the analysis to meteorological problems, the situation is as follows:

(1) The exponential terms in (38), corresponding to local disturbances only, are least affected by the replacement of the tropopause by a rigid boundary, and, since d is rather large compared with the vertical dimension of a barrier, are practically independent of d.

(2) For small values of F, the shorter lee-waves predicted by the theory will exist in the atmosphere. The error committed by imposing the upper rigid boundary is small in connexion with the shorter lee-waves.

(3) For the longer lee-waves, the error may be large.

(4) The theory will furnish no information at all on lee-waves extending to the stratosphere.

Because of the situation stated in (3) and (4), further work is necessary to determine more accurately the effect of the upper boundary. In the absence of better information, d should be taken to be the depth of the troposphere when applying the theory presented here (which does have the advantage that the amplitude of the motion treated does not have to be small).

5. Class (c): another class of waves of arbitrary amplitude, with possible application to atmospheric flows

For convenience, the coefficient A in (12c) will be denoted by α^2, so that α is imaginary if A is negative. The solution of (12c) is of the form

$$\Psi = \Psi_1(\eta)+\Psi_2(\xi,\eta), \tag{46}$$

in which Ψ_1 satisfies the equation

$$\frac{d^2\Psi_1}{d\eta^2}+\alpha^2\eta\Psi_1 = B, \tag{47}$$

and Ψ_2 satisfies the equation

$$\nabla^2\Psi_2+\alpha^2\eta\Psi_2 = 0. \tag{48}$$

The density variation now satisfies the equation $d\rho/d\Psi \propto \Psi$.

For $B = 0$ the solution of (47) is, aside from a constant factor,

$$\Psi_1(\eta) = \eta^{\frac{1}{2}}\{J_{\frac{1}{3}}(\tfrac{2}{3}\alpha\,\eta^{\frac{3}{2}})+CN_{\frac{1}{3}}(\tfrac{2}{3}\alpha\,\eta^{\frac{3}{2}})\} \equiv f(\eta). \tag{49}$$

For $B \neq 0$ the substitution

$$\Psi_1 = f(\eta)\,\phi(\eta)+K_1 \tag{50}$$

leads to

$$\Psi_1 = f\left[\int_0^{\eta}\frac{B\int_0^{\eta}f\,d\eta-\alpha^2K_1\int_0^{\eta}\eta f\,d\eta}{f^2}\,d\eta+K'\right]+K_1. \tag{51}$$

Since $f(0) = 0$, $\Psi_1 = K_1$ at $\eta = 0$. (It is no longer always permissible to take $\Psi = 0$ at $\eta = 0$, because in reaching (11c) Ψ is already assumed to have been modified by an additive constant if necessary.) The constant K' in (51) ensures that Ψ_1 can be adjusted to take any value K_2 at $\eta = 1$ provided $J_{\frac{1}{3}}(\frac{2}{3}\alpha) \neq 0$.

The solution of (48) is of the form

$$\Psi_2 = \sum_{n=1}^{\infty} A_n \exp\left[\pm (a_n \xi)\, \beta(\alpha, b_n, \eta)\right], \tag{52}$$

in which $b = -a^2/\alpha^{\frac{7}{3}}$, and

$$\beta(\alpha, b, \eta) = (\eta - b)^{\frac{1}{2}} \{N_{\frac{1}{3}}[\tfrac{2}{3}\alpha(-b)^{\frac{3}{2}}]\, J_{\frac{1}{3}}[\tfrac{2}{3}\alpha(\eta - b)^{\frac{3}{2}}] - J_{\frac{1}{3}}[\tfrac{2}{3}\alpha(-b)^{\frac{3}{2}}]\, N_{\frac{1}{3}}[\tfrac{2}{3}\alpha(\eta - b)^{\frac{3}{2}}]\}, \tag{53}$$

and b_n are the eigenvalues satisfying

$$\beta(\alpha, b, 1) = 0. \tag{54}$$

If a_n^2 is negative, (52) corresponds to a wave motion with periodic components.

Solutions Ψ_1 and Ψ_2, given in (51) and (52), can be used for application of Long's method or the alternative method proposed in § 4, in dealing with flow over a barrier. In fact, the entire development in § 4 can be repeated in a strictly similar manner for the new forms of Ψ_1 and Ψ_2, and the problem of flow into a sink treated by Yih (1958) can again be solved if the upstream condition is described by (51) and $|A|$ is sufficiently small. We shall not work out the details, which are tedious but straightforward, and shall content ourselves by pointing out that (51) again represents infinitely many upstream conditions, corresponding to the infinitely many sets of values for A, B, K', and K_1, one of which may sometimes be a better approximation to the actual wind distribution than one represented by (28).

This work has been done during the tenure of a Senior Post-Doctoral Fellowship of the National Science Foundation at the University of Cambridge, and the author wishes to express his appreciation to the Foundation for the opportunity to pursue uninterrupted research. The author is indebted to Dr G. K. Batchelor, F.R.S., for stimulation, criticism, and several useful discussions. It is also a pleasure to acknowledge the support of fundamental research on stratified flows by the Office of Ordnance Research in the past several years, without which the present work would have taken much longer to accomplish.

Note added in proof. While this paper was in the proof stage, its contents were presented at a seminar of the Institute of Meteorology of the University of Stockholm. At that time Prof. G. Benton kindly informed the author that some of the results presented had already been published by Prof. R. R. Long in a brief note ('Tractable models of steady-state stratified flow with shear', *Quart. J. Roy. Met. Soc.* **84**, 1958, pp. 159–61). Upon examining that note, I found that equation (28) and the upstream condition corresponding to equation (12*c*) had already been obtained by Long by an entirely different approach.

REFERENCES

Long, R. R. 1953 Some aspects of the flow of stratified fluids. I. A theoretical investigation. *Tellus*, **5**, 42–57.

Long, R. R. 1955 Some aspects of the flow of stratified fluids. III. Continuous density gradients. *Tellus*, **7**, 342–57.

Yih, C.-S. 1958 On the flow of a stratified fluid. *Proc. Third U.S. Nat. Congr. Appl. Mech.* pp. 857–61.

Yih, C.-S. 1960 Gravity waves in a stratified fluid. *J. Fluid Mech.* **8**, 481–508.